

Dyson-Schwinger loop equations of the two-matrix model: Eigenvalue correlations in quantum chaos

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We determine a set of Dyson-Schwinger equations or loop equations for a model of two coupled random matrices belonging to the orthogonal, unitary, or symplectic ensembles. In the large- N limit, the loop equations become closed algebraic equations, allowing us to obtain the correlations between the eigenvalues of the two matrices. The expression we obtain is valid near the center as well as the edge of the cut. In particular, this determines how the correlations between the eigenvalues of perturbed and unperturbed chaotic Hamiltonians depend upon the strength of the perturbation, and also the space and time dependence of density-density correlators of the Calogero-Sutherland-Moser model for three values of the coupling constant.

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I. INTRODUCTION

There is considerable recent interest in the connection between apparently irreconcilable fields, namely, chaotic quantum systems and exactly integrable many particle systems of a certain kind. Quantum chaotic systems are known to possess a certain universality in terms of their correlation functions [1]; this is the well known universality underlying the Wigner-Dyson random matrix theory. These universal correlation functions are known to be the ground state correlation functions of a many body system in one dimensional (1D) that is exactly integrable [2], the $1/r^2$ quantum many body system known as the Calogero, Sutherland, and Moser (CSM) system [3]. The recent work [4] relates the correlation functions of a perturbed quantum chaotic system to those of *time dependent* correlations of the CSM system. These connections have enriched both fields, and have led to new calculational schemes for objects of interest.

The CSM system may be defined via the Hamiltonian

$$H = - \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{\Omega^4}{4} \sum_{i=1}^N x_i^2 + \lambda(\lambda - 1) \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}. \quad (1)$$

Among the quantities of interest here are correlation functions $\langle \rho(x, 0) \rho(y, t) \rangle$, etc., involving the density of particles $\rho(x, t) = \sum_{i=1}^N \delta(x - x_i(t))$. The choice $\Omega = \sqrt{\pi^2 \lambda / N}$ leads to a normalization $\langle \rho(0, t) \rangle = 1$ in the center of the Wigner semicircle. The harmonic confining well is missing in the Sutherland model, where the

particles are confined to a ring.

In quantum chaos one is interested in the correlation between the eigenvalues E_i of the Hamiltonian H of a classically chaotic system, e.g., averages of the type $\langle \rho_H(E) \rho_H(E') \rangle$, where $\rho_H(E) \equiv \sum_{i=1}^N \delta(E - E_i)$ is the density of eigenvalues. These are described by treating H as an $N \times N$ random matrix with a Gaussian probability distribution, in the $N \rightarrow \infty$ limit. The correspondence between these eigenvalue correlators and the equal-time density correlators, e.g., $\langle \rho(x, 0) \rho(y, 0) \rangle$, of the CSM model is well known: the eigenvalue (connected) correlators of H in the three different ensembles, orthogonal, unitary, and symplectic, are identical to the CSM correlators with the values of the coupling constant $\lambda = \frac{1}{2}, 1, 2$, respectively, with the position coordinate x of the CSM system corresponding to the energy E of the chaotic Hamiltonian.

In this paper we consider the time dependent correlators $\langle \rho(x, 0) \rho(y, t) \rangle$ of the CSM model, as well as the eigenvalue correlations in quantum chaos which depend upon the strength of a perturbation parameter. The eigenvalue correlators are of the type $\langle \rho_{H_0}(E) \rho_H(E') \rangle$; H_0 is an unperturbed Hamiltonian, H is the perturbed Hamiltonian

$$H = H_0 \cos \Omega \phi + H_1 \frac{\sin \Omega \phi}{\Omega}, \quad (2)$$

where the perturbation H_1 is in the same universality class as H_0 (described by the same ensemble), and ϕ determines the strength of the perturbation. The above CSM and quantum chaos correlators are the same with a mapping that relates time to the strength of the perturbation: $t = -\ln[\cos(\Omega \phi)] / \Omega^2$ (in the large- N limit $\Omega \rightarrow 0$, $t \rightarrow \phi^2 / 2$).

This correspondence between the two systems is established [5] by mapping both onto a third system: the random matrix model. In particular, the connection is with the two-matrix model defined by the partition function

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$$Z = \int dA dB e^{-S} \quad (3)$$

where $S = N \text{Sp} [V(A) + V(B) - cAB]$ and $V(A) = \frac{1}{2} \mu A^2$. For $\lambda = \frac{1}{2}$, A and B are $N \times N$ real symmetric matrices (orthogonal ensemble), for $\lambda = 1$, A and B are $N \times N$ Hermitian matrices (unitary ensemble), and for $\lambda = 2$ $N \times N$ real self-dual quaternions (symplectic ensemble). $\text{Sp}(A)$ stands for $\text{Tr} A$ for $\lambda = \frac{1}{2}, 1$ and for $\frac{1}{2} \text{Tr} A$ for $\lambda = 2$. The parameters are related to those of (1) and (2) through $\mu = \frac{\Omega^2}{\sin^2 \Omega \phi}$ and $c = \frac{\Omega^2 \cos \Omega \phi}{\sin^2 \Omega \phi}$. Note that since $\Omega \sim O(1/\sqrt{N})$ μ and c are $O(1)$ when t and ϕ are $O(1)$.

In this model the connected density-density correlator is $\rho_{AB}(x, y) \equiv \langle \hat{\rho}_A(x) \hat{\rho}_B(y) \rangle_c$, where the density is defined as $\hat{\rho}_A(x) \equiv \frac{1}{N} \text{Tr} \delta(x - A)$, $\langle X \rangle \equiv \frac{1}{Z} \int dA dB e^{-S} X$, and the subscript c implies the connected part. In the limit $N \rightarrow \infty$ the correlator $\rho_{AB}(x, y)$ is related to the correlators mentioned in the above two systems with the following identifications. For quantum chaos, $\langle \rho_{H_0}(E_1) \rho_H(E_2) \rangle_c |_{\text{quantum chaos}} = N \rho_{AB}(E'_1, E'_2)$, with $E'_i = N^{-\frac{1}{2}} E_i$. For the CSM model, $\langle \rho(x, 0) \rho(y, t) \rangle_c |_{\text{CSM model}} = N \rho_{AB}(x', y')$, with $x' = N^{-\frac{1}{2}} x$ and $y' = N^{-\frac{1}{2}} y$.

Hence the problem of finding the correlators in quantum chaos or the CSM model is one of finding the correlator in the matrix model (3). Various different techniques [6,7,4,8,9] have been used to do this, with results that have various domains of validity. Here we develop another method of obtaining these correlators based on the method of loop equations (Dyson-Schwinger equations) generally applied in the context of QCD, noncritical string theory, and 2D quantum gravity, to obtain some additional results.

II. RESULTS: CORRELATORS AND LOOP EQUATIONS

In the large- N limit, the expectation value for the density is given by the well known Wigner semicircle law [10,1,11]

$$\langle \hat{\rho}_A(x) \rangle = \langle \hat{\rho}_B(x) \rangle = \frac{2}{\pi a^2} \sqrt{a^2 - x^2}, \quad |x| \leq a, \quad (4)$$

and $\langle \hat{\rho} \rangle = 0$ for $|x| \geq a$, where a , the ‘‘end point of the cut’’ is given by $a = (\frac{4\lambda\mu}{\mu^2 - c^2})^{\frac{1}{2}}$. Our result for the connected density-density correlator to leading order in $\frac{1}{N}$, valid over the entire cut, is

$$\begin{aligned} \rho_{AB}(x, y) = & -\frac{1}{4\pi^2 N^2} \frac{1}{\lambda a^2} \frac{1}{\cos \theta \cos \alpha} \\ & \times \left[\frac{1 + \cosh u \cos(\theta + \alpha)}{[\cosh u + \cos(\theta + \alpha)]^2} \right. \\ & \left. + \frac{1 - \cosh u \cos(\theta - \alpha)}{[\cosh u - \cos(\theta - \alpha)]^2} \right], \quad (5) \end{aligned}$$

where $u \equiv \ln(\frac{c}{\mu})$, $|x|, |y| < a$, and we have defined $\sin \theta = \frac{x}{a}$ and $\sin \alpha = \frac{y}{a}$. For $\lambda = 1$ (free fermions)

the above result was derived in [9] using methods different from ours. Our method, described below, is capable of generalization to $\lambda = \frac{1}{2}, 2$ (which corresponds to interacting fermions) with the above result.

We find the connected Green function $G_{AB}(z, w) \equiv \langle \hat{W}_A(z) \hat{W}_B(w) \rangle_c$, where $\hat{W}_A(z) \equiv \frac{1}{N} \text{Tr} \frac{1}{(z-A)}$, to leading order in $\frac{1}{N}$:

$$\begin{aligned} G_{AB}(z, w) = & \frac{1}{N^2} \frac{ca^2}{4\mu\lambda} \frac{1}{[1 - \frac{ca^2}{4\mu} W(z)W(w)]^2} \\ & \times \left[\frac{W(z)^2}{[1 - \frac{a^2}{4} W^2(z)]} \right] \left[\frac{W(w)^2}{[1 - \frac{a^2}{4} W^2(w)]} \right]. \quad (6) \end{aligned}$$

Here

$$W(z) = \langle \hat{W}_A(z) \rangle = \langle \hat{W}_B(z) \rangle = \frac{2}{a^2} [z - \sqrt{z^2 - a^2}]. \quad (7)$$

Equation (5) then follows from the identity

$$\begin{aligned} \rho_{AB}(x, y) = & -\frac{1}{4\pi^2} \lim_{\epsilon \rightarrow 0} [G_{AB}(x + i\epsilon, y + i\epsilon) \\ & + G_{AB}(x - i\epsilon, y - i\epsilon) - G_{AB}(x + i\epsilon, y - i\epsilon) \\ & - G_{AB}(x - i\epsilon, y + i\epsilon)]. \quad (8) \end{aligned}$$

We note that in the connected correlators (6) and (5), the dependence on λ appears only as an overall factor, and through the end point of the cut a .

The connected Green function $G_{AB}(z, w)$ is obtained as a solution of a set of closed algebraic equations relating the following five correlators of the two-matrix model: $W(z)$, $W^{(2)}(z, w) \equiv \langle \frac{1}{N} \text{Tr} (\frac{1}{(z-A)} \frac{1}{(w-B)}) \rangle$, $W_{A,1}^{(2)}(z) \equiv \langle \hat{W}_{A,1}(z) \rangle \equiv \langle \frac{1}{N} \text{Tr} (\frac{1}{(z-A)} B) \rangle$, $\langle \hat{W}_{A,1}(z) \hat{W}_B(w) \rangle$, and $G_{AB}(z, w)$. The closed set of algebraic equations is

$$0 = -\mu[zW(z) - 1] + cW_{A,1}(z) + \lambda W(z)^2, \quad (9)$$

$$0 = -\mu W_{A,1}(z) + c[zW(z) - 1], \quad (10)$$

$$0 = [-\mu z + cw + \lambda W(z)]W^{(2)}(z, w) + \mu W(w) - cW(z), \quad (11)$$

$$0 = [2\lambda W(z) - \mu z]G_{AB}(z, w) + c\langle \hat{W}_{A,1}(z) \hat{W}_B(w) \rangle_c, \quad (12)$$

$$\begin{aligned} 0 = & -\mu \langle \hat{W}_{A,1}(z) \hat{W}_B(w) \rangle_c + czG_{AB}(z, w) \\ & - \frac{1}{N^2} \frac{\partial}{\partial w} W^{(2)}(z, w). \quad (13) \end{aligned}$$

In Eqs. (9)–(13) only terms of the same order in the $\frac{1}{N}$ expansion have been exhibited on the right-hand side (RHS); higher order terms in $\frac{1}{N}$ have been suppressed. It is only with this suppression (valid in the large- N limit) that one gets closed equations. Such equations were originally obtained for the one-matrix model in the context of large- N QCD [12,13] and string theory [14,15], where

they are referred to as “loop equations” since an insertion of $\hat{W}_A(z)$ has the geometric interpretation of creating a hole or loop on the string world sheet. Loop equations for the two-matrix model have also been discussed earlier [16,17] in the context of 2D quantum gravity. However, they do not involve the correlator G_{AB} which is relevant for quantum chaos and the CSM system, and are restricted to the case $\lambda = 1$.

(9) and (10) give (7) [from which follows (4)], and (11) gives [17] $W^{(2)}(z, w) = \frac{\mu W(w) - cW(z)}{\mu z - cw - \lambda W(z)}$. Finally, (12) and (13) give $G_{AB}(z, w) = -\frac{1}{N^2} \frac{c}{\mu} \frac{1}{2\lambda} \frac{1}{[2z/a^2 - W(z)]} \frac{\partial}{\partial w} W^{(2)}(z, w)$ which can be shown to be equal to (6) with some algebra.

III. DERIVATION OF THE LOOP EQUATIONS

The above set of closed equations is obtained by starting from a set of “Ward identities.” For example, to derive (9) consider the identity

$$0 = \int dA \, dB \, \frac{\partial}{\partial A_{ij}} [e^{-S}(A^n)_{ij}], \quad (14)$$

where i and j are not summed over. For $\lambda = \frac{1}{2}$ (A and B are real symmetric matrices), we have

$$\frac{\partial A_{ij}}{\partial A_{kl}} = \frac{1}{g_{ij}} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (15)$$

where [1] $g_{ij} = \delta_{ij} + 1$. This implies that

$$\begin{aligned} \frac{\partial}{\partial A_{ij}} (A^n)_{ij} &= \frac{1}{g_{ij}} \left[\sum_{k=0}^{n-1} (A^k)_{jj} (A^{n-1-k})_{ii} + 2\delta_{ij} (A^{n-1})_{ij} \right. \\ &\quad \left. + \sum_{k=1}^{n-2} (A^k)_{ij} (A^{n-1-k})_{ij} \right] \quad (n \geq 2), \quad (16) \end{aligned}$$

and

$$\frac{\partial}{\partial A_{ij}} \text{Tr} f(A) = \frac{2}{g_{ij}} [f'(A)]_{ji}. \quad (17)$$

Using (16) and (17) in (14), then multiplying by $\frac{g_{ij}}{2N^2}$ and summing over i and j , we get

$$\begin{aligned} 0 &= -\mu \left\langle \frac{1}{N} \text{Tr} A^{n+1} \right\rangle + c \left\langle \frac{1}{N} \text{Tr} A^n B \right\rangle \\ &\quad + \frac{1}{2} \sum_{k=0}^{n-1} \left\langle \frac{1}{N} \text{Tr} A^k \frac{1}{N} \text{Tr} A^{n-1-k} \right\rangle \\ &\quad + \frac{n}{2N} \left\langle \frac{1}{N} \text{Tr} A^{n-1} \right\rangle. \quad (18) \end{aligned}$$

Dividing (18) by $\frac{1}{z^{n+1}}$ and summing over n from 0 to ∞ gives

$$\begin{aligned} 0 &= -\mu [zW(z) - 1] + cW_{A,1}(z) + \frac{1}{2} \langle \hat{W}_A(z) \hat{W}_A(z) \rangle \\ &\quad - \frac{1}{2N} \frac{\partial}{\partial z} W(z). \quad (19) \end{aligned}$$

Now decompose $\langle \hat{W}_A(z) \hat{W}_A(z) \rangle$ into its disconnected and connected parts: $\langle \hat{W}_A(z)^2 \rangle = \langle \hat{W}_A(z) \rangle^2 + \langle \hat{W}_A(z) \hat{W}_A(z) \rangle_c$. Recognizing that $\langle \hat{W}_A(z) \rangle = W(z)$ is $O(1)$ and $\langle \hat{W}_A(z) \hat{W}_A(z) \rangle_c$ is $O(\frac{1}{N^2})$, and suppressing the latter as well as the last term in (19), gives (9).

Following a similar procedure Eqs. (10), (11), (12), and (13) follow, respectively, from the identities

$$0 = \int dA \, dB \, \frac{\partial}{\partial B_{ij}} [e^{-S}(A^n)_{ij}], \quad (20)$$

$$0 = \int dA \, dB \, \frac{\partial}{\partial A_{ij}} [e^{-S}(A^n B^m)_{ij}], \quad (21)$$

$$0 = \int dA \, dB \, \frac{\partial}{\partial A_{ij}} [e^{-S}(A^n)_{ij} \hat{W}_B(w)], \quad (22)$$

$$0 = \int dA \, dB \, \frac{\partial}{\partial B_{ij}} \{e^{-S}[(z-A)^{-1}]_{ij} \hat{W}_B(w)\}. \quad (23)$$

Note that the terms on the RHS of Eqs. (12) and (13) are all of $O(\frac{1}{N^2})$, while those in (9)–(11) are $O(1)$. The reason this happens is that all $O(1)$ terms in (22) and (23) cancel. For example, starting from (22) and following the steps described above one obtains an intermediate equation

$$\begin{aligned} 0 &= -\mu \langle [z\hat{W}_A(z) - 1] \hat{W}_B(w) \rangle + c \langle \hat{W}_{A,1}(z) \hat{W}_B(w) \rangle \\ &\quad + \frac{1}{2} \langle \hat{W}_A(z) \hat{W}_A(z) \hat{W}_B(w) \rangle - \frac{1}{2N} \frac{\partial}{\partial z} \langle \hat{W}_A(z) \hat{W}_B(w) \rangle. \quad (24) \end{aligned}$$

When this equation is separated into disconnected and connected parts using $\langle fg \rangle = \langle f \rangle \langle g \rangle + \langle fg \rangle_c$, $\langle fgh \rangle = \langle f \rangle \langle g \rangle \langle h \rangle + \langle fg \rangle_c \langle h \rangle + \langle f \rangle \langle gh \rangle_c + \langle g \rangle \langle fh \rangle_c + \langle fgh \rangle_c$, one sees that there is a cancellation of terms that come arranged in a form that is proportional to the RHS of (19). This leaves us with

$$\begin{aligned} 0 &= [W(z) - \mu z] G_{AB}(z, w) + c \langle \hat{W}_{A,1}(z) \hat{W}_B(w) \rangle_c \\ &\quad + \frac{1}{2} \langle \hat{W}_A(z)^2 \hat{W}_B(w) \rangle_c - \frac{1}{2N} \frac{\partial}{\partial z} G_{AB}(z, w), \quad (25) \end{aligned}$$

which is the same as (12) when the last two terms, which are $O(\frac{1}{N^4})$ and $O(\frac{1}{N^5})$, respectively, are suppressed.

The identities for the unitary and symplectic ensemble are arrived at by following the same sequence of steps but with the following relations for the derivatives in place of (16) and (17). For $\lambda = 1$, since A is a Hermitian matrix, one uses $\frac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik} \delta_{jl}$, which implies $\frac{\partial}{\partial A_{ij}} (A^n)_{ij} = \sum_{k=0}^{n-1} (A^k)_{jj} (A^{n-1-k})_{ii}$ for $(n \geq 1)$, and $\frac{\partial}{\partial A_{ij}} \text{Tr} f(A) = f'(A)_{ji}$. For $\lambda = 2$ (symplectic ensemble), $A_{ij} = A_{ij}^{(0)} e^{(0)} + \sum_{\alpha=1}^3 A_{ij}^{(\alpha)} e^{(\alpha)}$, where $(e^{(0)}, e^{(1)}, e^{(2)}, e^{(3)}) = (I, i\sigma^{(3)}, i\sigma^{(2)}, i\sigma^{(1)})$, I is the 2×2 unit matrix, and $\sigma^{(\alpha)}$ are the Pauli matrices. In this case the independent variables are the real symmetric $N \times N$ matrix $A^{(0)}$ and the real antisymmetric $N \times N$ matrices

$A^{(\alpha)}$. Then, $\text{Sp } A = \frac{1}{2} \text{Tr } A = \sum_{i=1}^N A_{ii}^{(0)}$, and one finds

$$\frac{\partial}{\partial A_{ij}^{(0)}} (A^n)_{ij} = \frac{e^{(0)}}{g_{ij}} \left[\sum_{k=0}^{n-1} (A^k)_{jj} (A^{n-1-k})_{ii} + 2\delta_{ij} (A^{n-1})_{ij} + \sum_{k=1}^{n-2} (A^k)_{ij} (A^{n-1-k})_{ij} \right] \quad (n \geq 2), \quad (26)$$

$$\frac{\partial}{\partial A_{ij}^{(0)}} \text{tr} f(A) = \frac{e^{(0)}}{g_{ij}} [f'(A)_{ji} + f'(A)_{ij}], \quad (27)$$

$$\frac{\partial}{\partial A_{ij}^{(\alpha)}} (A^n)_{ij} = e^{(\alpha)} \left[\sum_{k=0}^{n-1} (A^k)_{jj} (A^{n-1-k})_{ii} - 2\delta_{ij} (A^{n-1})_{ij} - \sum_{k=1}^{n-2} (A^k)_{ij} (A^{n-1-k})_{ij} \right], \quad (28)$$

and

$$\frac{\partial}{\partial A_{ij}^{(\alpha)}} \text{tr} f(A) = e^{(\alpha)} [f'(A)_{ji} - f'(A)_{ij}], \quad (29)$$

where $\text{tr } A \equiv \sum_{k=1}^N A_{kk}$ is a 2×2 matrix. Using these identities in (14) and (20)–(23) gives the loop equations (9)–(13). This completes the derivation of (6) for all three ensembles.

IV. CONCLUSIONS AND DISCUSSION

To sum up, we have presented a method for calculating correlators in random matrix models based on loop equations which leads to the result (5) for the density-density correlator. This method treats all three ensembles on the same footing because the procedure for deriving the loop equations remains the same. The only distinction between the ensembles enters at the level of counting the degrees of freedom, that is, through equations such as (15) [18].

The result (5) averages over eigenvalues around x and y independently [19]. While this independent averaging over the positions of x and y does lose information about an oscillating piece of the correlation function, namely, the $2k_F$ piece, for many physical applications only the smoothed results are needed.

As mentioned in the Introduction, a number of methods for obtaining density-density correlators in the two-matrix model exist in the literature. In particular, the supersymmetry technique [6] and methods of Ref. [7] can also provide information about the oscillating piece. However, to the best of our knowledge, the formula (5) for the smoothed correlator does not appear in the literature except for the case of the unitary ensemble ($\lambda = 1$) [9].

We now discuss the universality of this formula. It is satisfying that for all three ensembles the result is the same, up to an overall factor of λ and the dependence of the edge of the cut on λ . However, the formula has been derived above only for Gaussian ensembles,

$V(A) = \frac{1}{2} \mu A^2$. The question is whether it holds for more general ensembles. We remark here that in two physically different limits (5) is universal. First, near the center of the Wigner semicircle and $\phi^2 \ll O(N)$, (5) reduces to [4,8]

$$\langle \rho(x, 0) \rho(y, t) \rangle_c = -\frac{1}{2\pi^2 \lambda^2} \frac{[(x-y)^2 - 4\pi^2 \lambda^2 t^2]}{[(x-y)^2 + 4\pi^2 \lambda^2 t^2]^2}. \quad (30)$$

This identifies the sound velocity as $v_s = 2\pi\lambda$ in the CSM model. The RHS of (30) also equals $\langle \rho_{H_0}(x) \rho_H(y) \rangle_c$ for quantum chaos. Though an analytic proof does not yet exist, there is substantial evidence [4,5] that (30) is universal. Second, in the ‘‘equal-time’’ limit ($t = 0$ or $u = 1$), (5) reduces to

$$\rho_{AB}(x, y) = -\frac{1}{2\pi^2 N^2 \lambda^2} \frac{1}{(x-y)^2} \frac{a^2 - xy}{\sqrt{(a^2 - x^2)(a^2 - y^2)}}. \quad (31)$$

This expression was obtained in Ref. [20] for Gaussian ensembles of a single random matrix, and proven to be universal for all even polynomial potentials in the unitary ensemble [21]. The proof of universality has been extended to the other ensembles also [22]. Thus the equal-time limit of (5) is known to be universal for all x and y , near the center as well as near the edge of the cut. For further observations regarding universality see [9].

Using the method of loop equations we have obtained the expression [analogous to the one above Eq. (14)] for $G_{AB}(z, w)$ for an arbitrary polynomial potential of degree m in terms of $W(z)$. In order to establish universality it remains to check that this expression leads to (5) using the fact that $W(z)$ is itself a solution of an m th degree polynomial equation [17]. The number of equations needed for closure increases with the degree because new types of correlators get coupled into the existing Eqs. (9)–(13). For example, when $V(A)$ is cubic, the correlators $\langle \hat{W}_2(z) \rangle$ and $\langle \hat{W}_2(z) \hat{W}_B(w) \rangle_c$ [where $\hat{W}_2(z) \equiv \frac{1}{N} \text{tr}(\frac{1}{z-A} B^2)$] get coupled. The equations also contain a finite number (depending on m) of other unknown quantities such as $\langle \text{tr} A \rangle$ and $\langle \text{tr} A^2 \rangle$, but these are all self-consistently determined from the loop equations and the analyticity properties of $W(z)$. The procedure for deriving the additional equations remains similar to the one discussed above. These results as well as higher order correlators and correlators for crossover to different ensembles will be presented elsewhere.

We remark that (5) or (6) is valid for all x and y including near the edge of the semicircle and for all time. Thus it may have a larger applicability than for the CSM model and quantum chaos, e.g., in quantum gravity and string theory, or in conductance fluctuations of mesoscopic conductors where behavior near the edge of the cut is particularly relevant [22].

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- [1] F. Dyson, *J. Math. Phys.* **3**, 140 (1962); **3**, 157 (1962); **3**, 166 (1962); **3**, 1191 (1962); *Commun. Math. Phys.* **19**, 235 (1970).
- [2] B. Sutherland, *J. Math. Phys.* **12**, 246; **12**, 251 (1971).
- [3] F. Calogero, *J. Math. Phys.* **10**, 2197 (1969); J. Moser, *Adv. Math.* **16**, 197 (1975).
- [4] A. Szafer and B. L. Altschuler, *Phys. Rev. Lett.* **70**, 587 (1993); B. D. Simons and B. L. Altschuler, *ibid.* **70**, 4063 (1993); B. D. Simons, P. A. Lee, and B. L. Altschuler, *ibid.* **70**, 4122 (1993); B. D. Simons, A. Szafer, and B. L. Altschuler, *Pis'ma Zh. Eksp. Teor. Fiz.* **57**, 268 (1993) [*JETP Lett.* **57**, 276 (1993)].
- [5] O. Narayan and B. S. Shastry, *Phys. Rev. Lett.* **71**, 13 (1993); B. S. Shastry, in *Correlation Effects in Low Dimensional Electron Systems*, edited by A. Okiji and N. Kawakami (Springer, Berlin, 1994).
- [6] K. B. Efetov, *Adv. Phys.* **32**, 53 (1983).
- [7] M. L. Mehta and A. Pandey, *J. Phys. A* **16**, 2655 (1983); A. Pandey and M. L. Mehta, *Commun. Math. Phys.* **87**, 449 (1983); A. Pandey, Jawaharlal Nehru University report, 1994 (unpublished).
- [8] C. W. J. Beenakker, *Phys. Rev. Lett.* **70**, 4126 (1993).
- [9] E. Brezin and A. Zee, *Nucl. Phys.* **B424**, (FS) 435 (1994); *Phys. Rev. E* **49**, 2588 (1994).
- [10] E. Wigner, in *Canadian Mathematical Conference Proceedings* (University of Toronto Press, Toronto, 1957), reproduced in *Statistical Theories of Spectra: Fluctuations*, edited by C. E. Porter (Academic, New York, 1965).
- [11] M. L. Mehta, *Random Matrices*, 2nd ed. (Academic, San Diego, 1991).
- [12] S. R. Wadia, *Phys. Rev. D* **24**, 970 (1981).
- [13] A. A. Migdal, *Phys. Rep.* **102**, 199 (1983).
- [14] V. A. Kazakov, *Mod. Phys. Lett. A* **4**, 2125 (1989).
- [15] F. David, *Mod. Phys. Lett. A* **5**, 1019 (1990).
- [16] E. Gava and K. S. Narain, *Phys. Lett. B* **263**, 213 (1991).
- [17] M. Staudacher, *Phys. Lett. B* **305**, 332 (1993).
- [18] We thank N. Mukunda for this observation.
- [19] The reason for this (already indicated in [9]) is the following: for μ , c , and a of $O(1)$, the average distance between eigenvalues is $O(1/N)$ [$O(1/\sqrt{N})$ if $a \sim O(\sqrt{N})$]. When the large- N limit is taken (for closure of the loop equations) the imaginary parts of z and w are typically $O(1)$. Thus, since the distance of z and w from the real axis is much larger than the typical separation between eigenvalues, both z and w in (6) experience only an averaged effect of the eigenvalues. Thus one automatically gets a smoothed-out result when the limit (8) is taken.
- [20] J. B. French, P. A. Mello, and A. Pandey, *Ann. Phys. (N.Y.)* **113**, 277 (1978); for a review, see T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, *Rev. Mod. Phys.* **53**, 385 (1981).
- [21] E. Brezin and A. Zee, *Nucl. Phys.* **B402**, 613 (1993); *C. R. Acad. Sci.* **317**, 735 (1993).
- [22] C. W. J. Beenakker, *Nucl. Phys.* **B422**, (FS) 515 (1994); *Phys. Rev. Lett.* **70**, 1155 (1993).